Study of a Noise Amplification Mechanism in Gryrotrons

KUAN REN CHEN AND KWO RAY CHU, SENIOR MEMBER, IEEE

Abstract—The electrostatic cyclotron instability driven by a relativistic mechanism is analyzed with a small-signal theory and is shown to be a significant noise amplification mechanism in gyrotrons. The mode is localized in the electrotron beam, with a growth rate strongly dependent on the wavelength and electron density but insensitive to the velocity spread.

I. Introduction

THE ELECTRON CYCLOTRON frequency is inversely proportional to the relativistic mass of the electron. When electrons are gyrating in synchronism with a wave whose electric field is polarized in the plane of the gyration, some electrons will gain energy and rotate slower while others lose energy and rotate faster. The resulting bunching in the electron cyclotron phase space can lead to amplification of the wave. If the wave is electromagnetic (e.g., TE mode of a waveguide), the amplification is known as the electron cyclotron maser instability. If the wave is the electrostatic cyclotron modes of the electron medium, the amplification has been referred to as the relativistic cyclotron instability [1], the space charge instability [2], and the (unstable) Bernstein modes [3] in various studies. The cyclotron maser instability has been investigated extensively in recent years in the development of gyrotrons. In comparison, the electrostatic cyclotron instability (as we call it here in order to distinguish from the electromagnetic cyclotron instability) has received much less attention. In the context of gyrotrons, Charbit *et al.* [2] pointed out that the space-charge instability might lead to noise amplification in gyrotrons. Hirshfield [3] pointed out that the unstable Bernstein modes could be coupled to the electromagnetic circuit wave to generate radiation at high cyclotron harmonics.

In this paper, we present a detailed study of the electrostatic cyclotron instability on the basis of a model suitable for gyrotrons. Its significance on noise amplification in gyrotrons is emphasized.

The noise level in a gyrotron amplifier is believed to be high because gyrotrons commonly use a temperature-limited magnetron injection gun. High noise figures have indeed been measured in a gyro-TWT experiment [4]. Some contemplated applications of gyrotrons (e.g., as a high-power millimeter-wave source for deep space communication) require very low noise levels. Gyrotrons under consideration for these applications are usually amplifiers with long interaction sections. Conceivably, if noise is amplified in the interaction section, it could become a serious problem. The present analysis finds that the initially large noise level in the electron beam can be further amplified at a rate comparable to the RF signal amplification in a gyro-TWT, thus reinforcing the concern regarding excessive noise in gyrotrons. It is also shown (Section V) that these localized modes have little, if any, coupling to the circuit and, hence, may not be easily detected or stabilized.

Previous studies of the electrostatic cyclotron instability were motivated by different considerations and were pursued independently under different assumptions. As a result, the instability has been referred to by different terms, although they all involve the same basic mechanism. An interesting observation made in the course of our study (Section VII) is that some seemingly unconnected results in these works are in fact different limits of a general result (derived in Section III).

The dispersion relation for an infinite plasma is derived in Section II and analyzed in Sections III and IV for electron distribution functions appropriate for gyrotrons. It is shown in numerical plots and analytical approximations that the growth rate of the instability is strongly influenced by the effects of finite-electron Larmor radius and plasma density, but is weakly dependent on the electron velocity spread. The relativistic origin of the instability is algebraically traced in the dispersion relation, and physical interpretations of the solutions are given. A slab model with sharp boundaries is considered in Section V. It is found that the effect of finite geometry leads to discrete wavenumbers for the most unstable modes, while these modes are completely localized within the plasma slab. In Section VI, the results are applied to two gyro-TWT experiments, indicating significant noise amplification in both cases. Section VII compares the present work with some related works. In the last Section, we present a brief summary of the analytical approach and, from the perspective of the idealized model in Section V, comment on the generation of electromagnetic radiation through the electrostatic cyclotron instability.

II. Model, Assumptions, and Dispersion Relation

We are concerned with high-frequency waves in a neutral, uniform, and infinite plasma immersed in an external magnetic field $B_0\underset{\sim}{e}_z$. Since the medium is uniform and isotopic in the $x-y$ plane, the spatial and temporal dependence of the wave can be written as $\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}-i\omega t)$ with

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The authors are with the Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300, R.O C.

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$\underset{\sim}{k} = k_x \underset{\sim}{e}_x + k_z \underset{\sim}{e}_z$. If k is sufficiently large, some modes will become nearly electrostatic (i.e., $\nabla \times \underset{\sim}{E} \approx 0$). The electrostatic cyclotron mode, for example, is obtained under the condition

$$k_x^2 c^2 \gg \omega_{pe}^2 \tag{1}$$

where ω_{pe} is the plasma frequency.

Under the electrostatic approximation, the wave electric field is given by the electrostatic potential ϕ

$$\underset{\sim}{E} = -\nabla\phi \tag{2}$$

and ϕ satisfies the Poisson's equation

$$\nabla^2 \phi = -4\pi\rho. \tag{3}$$

The charge density can be written in terms of the current density $\underset{\sim}{J}$ through the equation of continuity

$$\rho = \frac{1}{\omega} \underset{\sim}{k} \cdot \underset{\sim}{J}. \tag{4}$$

Writing $\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$ where $\underset{\approx}{\sigma}$ is the electron conductivity tensor, we obtain from (2) and (4)

$$\rho = \frac{-i}{\omega} \underset{\sim}{k} \cdot \underset{\approx}{\sigma} \cdot \underset{\sim}{k}\phi. \tag{5}$$

Substitution of (5) in (3) gives

$$\underset{\sim}{k} \cdot \underset{\approx}{\epsilon} \cdot \underset{\sim}{k}\phi = 0 \tag{6}$$

where

$$\underset{\approx}{\epsilon} = \underset{\approx}{I} + \frac{4\pi i}{\omega} \underset{\approx}{\sigma} \tag{7}$$

is the plasma dielectric tensor and $\underset{\approx}{I}$ in (7) is a unit dyadic. Thus, the dispersion relation is

$$\underset{\sim}{k} \cdot \underset{\approx}{\epsilon} \cdot \underset{\sim}{k} = 0. \tag{8}$$

Equation (8) is a well-known expression of the general electrostatic dispersion relation [5].

With $\underset{\sim}{k}$ written as $k_x \underset{\sim}{e}_x + k_z \underset{\sim}{e}_z$, (8) reduces to

$$k_x^2 \epsilon_{xx} + 2k_x k_z \epsilon_{xz} + k_z^2 \epsilon_{zz} = 0 \tag{9}$$

where the Onsager relation $\epsilon_{xz} = \epsilon_{zx}$ has been substituted.

The dielectric tensor for a magnetized plasma is calculated from the linearized relativistic Vlasov equation and the Maxwell equations. The results have been documented in the literature [6, pp. 229], [7] and are written below without derivation:

$$\epsilon_{xx} = 1 - \frac{2m_e^2 \omega_{pe}^2 \Omega_e^2}{k_x^2 \omega} \sum_{n=-\infty}^{\infty} n^2 \left\langle J_n^2(k_x r_L) \cdot \left[\frac{\partial f_0}{\partial p_\perp^2}\left(1 - \frac{k_z p_z}{\gamma m_e \omega}\right) + \frac{k_z p_z}{\gamma m_e \omega} \frac{\partial f_0}{\partial p_z^2} \right] \right\rangle \tag{10a}$$

$$\epsilon_{xz} = -\frac{2m_e \omega_{pe}^2 \Omega_e}{k_x \omega} \sum_{n=-\infty}^{\infty} n \left\langle p_z J_n^2(k_x r_L) \cdot \left[\frac{\partial f_0}{\partial p_z^2} - \frac{n\Omega_e}{\omega}\left(\frac{\partial}{\partial p_z^2} - \frac{\partial}{\partial p_\perp^2} \right) \right] \right\rangle \tag{10b}$$

$$\epsilon_{zz} = 1 - \frac{2\omega_{pe}^2}{\omega} \sum_{n=-\infty}^{\infty} \left\langle p_z^2 J_n^2(k_x r_L) \cdot \left[\frac{\partial f_0}{\partial p_z^2} - \frac{n\Omega_e}{\omega}\left(\frac{\partial}{\partial p_z^2} - \frac{\partial}{\partial p_\perp^2} \right) f_0 \right] \right\rangle \tag{10c}$$

where

$$\langle \cdots \rangle \equiv 2\pi \int_0^\infty p_\perp \, dp_\perp \int_{-\infty}^{\infty} dp_z \frac{(\cdots)}{k_z p_z / m_e + n\Omega_e - \gamma\omega}.$$

In (10), $\omega_{pe} = (4\pi n_e e^2/m_e)^{1/2}$ is the rest mass plasma frequency, $\Omega_e = eB_0/m_e c$ is the rest mass electron cyclotron frequency, f_0 is the equilibrium distribution function, $r_L = p_\perp / m_e \Omega_e$ is the electron Larmor radius, $\gamma = [1 + (p_\perp^2 + p_z^2)/m_e^2 c^2]^{1/2}$ is the relativistic factor, $J_n(x)$ is the ordinary Bessel function of order n, and n is a cyclotron harmonic number.

Substitution of (10) in (9) results in the dispersion relation

$$k^2 - \frac{2\pi\omega_{pe}^2}{\omega} \int_0^\infty dp_\perp \int_{-\infty}^{\infty} dp_z \sum_{n=-\infty}^{\infty} \frac{J_n^2(k_x r_L)}{k_z p_z/m_e + n\Omega_e - \gamma\omega} \cdot \left\{ m_e^2 \frac{\partial f_0}{\partial p_\perp} \left[n^2 \Omega_e^2 \left(1 + \frac{k_z p_z}{\gamma m_e \omega}\right) + \frac{n k_z^2 p_z^2 \Omega_e}{\gamma m_e^2 \omega} \right] + m_e k_z p_\perp \frac{\partial f_0}{\partial p_z} \left[2n\Omega_e - \frac{n^2 \Omega_e^2}{\gamma\omega} + \frac{k_z p_z}{m_e}\left(1 - \frac{n\Omega_e}{\gamma\omega}\right) \right] \right\} = 0. \tag{11}$$

Up to this point, one could recover the nonrelativistic dispersion relation by simply replacing $\underset{\sim}{p}$ with $m_e \underset{\sim}{v}$ and γ with unity in (11). Here, the main purpose of the relativistic formalism is not merely to account for the quantitative difference between γ and unity. Rather, it is the weak dependence of the relativistic mass γm_e on the electron energy that gives rise to electron bunching in the cyclotron phase space and consequently drives the instability. This point becomes clear when we integrate (11) by parts over $p_\perp$ and p_z to eliminate the derivatives of f_0. After some algebra, we obtain

$$k^2 + 2\pi\omega_{pe}^2 \sum_{n=-\infty}^{\infty} \int_0^\infty p_\perp \, dp_\perp \int_{-\infty}^{\infty} dp_z \frac{f_0}{\gamma} \cdot \left\{ \frac{\gamma k_x^2}{2\Omega_e (k_z v_z + n\Omega_e/\gamma - \omega)} \cdot \left[\frac{n\Omega_e}{\gamma\omega}\left(1 + \frac{k_z v_z}{\omega}\right) + \frac{k_z^2 v_z^2}{\omega^2} \right] \left[J_{n-1}^2(k_x r_L) - J_{n+1}^2(k_x r_L) \right] + \frac{J_n^2(k_x r_L)}{(k_z v_z + n\Omega_e/\gamma - \omega)^2} \cdot \left[\underline{\frac{n^2 \Omega_e^2}{\gamma^2 c^2}} - k_z^2 \left(1 - \underline{\frac{2n\Omega_e v_z}{\gamma k_z c^2}} - \underline{\frac{v_z^2}{c^2}} \right) \right] \right\} = 0. \tag{12}$$

The underlined terms in (12) are due to differentiations of γ with respect to $p_\perp$ and p_z when carrying out the integration by parts of (11). These terms would disappear in the nonrelativistic limit ($c \to \infty$). It will be seen that the first of these terms is the origin of the instability.

We note in passing that, in the nonrelativistic limit, the well-known dispersion relations of the Bernstein modes and the Langmuir waves can be recovered from (12) by letting $k_z = 0$ and $k_x = 0$, respectively.

III. Monoenergetic Electrons without Velocity Spread

We shall consider modes that have

$$k_x \gg k_z \tag{13}$$

since these modes are most likely to persist even in the presence of a large velocity spread (see Section IV). Under condition (13), the solutions of (12) are essentially the Bernstein modes with frequencies in the neighborhood of the harmonics of the electron cyclotron frequency. Under the assumption

$$\frac{\omega}{k_z} \gg v_z \tag{14}$$

the last two of the underlined terms in (12) are negligible compared with the first underlined term. Assumption (14) will subsequently be justified when we show that the most unstable modes are those with $k_z = 0$ (see (21b)).

In this section, we specialize to the following distribution function:

$$f_0 = \frac{1}{2\pi p_\perp}\delta(p_\perp - p_{\perp 0})\delta(p_z - p_{z0}) \tag{15}$$

which represents a uniform distribution of monoenergetic electrons without a velocity spread.

Substituting (15) in (13) and applying assumption (14), we obtain

$$k^2 - \frac{k_z^2 \pi_{pe}^2 J_0^2(k_x r_{L0})}{(\omega - k_z v_{z0})^2} + \pi_{pe}^2 \sum_{n=1}^{\infty} \left\{ \frac{n k_x^2}{n^2\Omega_c^2 - (\omega - k_z v_{z0})^2} \cdot \left[J_{n-1}^2(k_x r_{L0}) - J_{n+1}^2(k_x r_{L0})\right] + \frac{2\left[n^2\Omega_c^2 + (\omega - k_z v_{z0})^2\right]}{\left[n^2\Omega_c^2 - (\omega - k_z v_{z0})^2\right]^2} J_n^2(k_x r_{L0}) \cdot \left(-k_z^2 + \underline{\frac{n^2\Omega_c^2}{c^2}} \right) \right\} = 0 \tag{16}$$

where $v_{z0} = p_{z0}/\gamma_0 m_e$, $r_{L0} = p_{\perp 0}/m_e\Omega_e$, $\Omega_c = \Omega_e/\gamma_0$, $\pi_{pe}^2 = \omega_{pe}^2/\gamma_0$, and $\gamma_0 = [1 + (p_{\perp 0}^2 + p_{z0}^2)/m_e^2 c^2]^{1/2}$. Note that π_{pe} and Ω_c are the relativistic plasma frequency and relativistic cyclotron frequency, respectively. Again, the relativistic term is underlined in (16).

Instability occurs when the Doppler shifted wave frequency $(\omega - k_z v_{z0})$ matches a cyclotron harmonic frequency $(n\Omega_c)$. Hence, (16) is dominated by the resonant harmonic term. By neglecting the nonresonant terms, (16) reduces to

$$1 - \frac{\pi_{pe}^2 V_n}{(\omega - k_z v_{z0})^2 - n^2\Omega_c^2} + \frac{2\pi_{pe}^2\left[(\omega - k_z v_{z0})^2 + n^2\Omega_c^2\right] U_n}{\left[(\omega - k_z v_{z0})^2 - n^2\Omega_c^2\right]^2} = 0 \tag{17}$$

where

$$V_n \equiv n\left[J_{n-1}^2(k_x r_{L0}) - J_{n+1}^2(k_x r_{L0})\right] k_x^2/k^2 \tag{18a}$$

and

$$U_n \equiv J_n^2(k_x r_{L0})\left(n^2\Omega_c^2/c^2 - k_z^2\right)/k^2. \tag{18b}$$

With the substitution $W = (\omega - k_z v_{z0})^2 - n^2\Omega_c^2$, equation (17) can be rewritten

$$W^2 - \pi_{pe}^2(V_n - 2U_n)W + 4n^2\pi_{pe}^2\Omega_c^2 U_n = 0 \tag{19}$$

and readily solved to give

$$W = (\omega - k_z v_{z0})^2 - n^2\Omega_c^2 = \frac{1}{2}\pi_{pe}^2(V_n - 2U_n) \pm 2n\Omega_c \cdot \pi_{pe}\left[\frac{\pi_{pe}^2}{16n^2\Omega_c^2}(V_n - 2U_n)^2 - U_n\right]^{1/2}.$$

Thus, the condition for ω to have complex solutions is

$$\pi_{pe}^2 < \frac{16n^2\Omega_c^2 U_n}{(V_n - 2U_n)^2}. \tag{20}$$

Under condition (20) and the assumption $\omega_i \ll \omega_r$, the unstable solution is given by

$$\omega_r \simeq k_z v_{z0} + n\Omega_c\left[1 + \frac{\pi_{pe}^2}{2n^2\Omega_c^2}(V_n - 2U_n)\right]^{1/2} \tag{21a}$$

$$\omega_i \simeq \pi_{pe}\left[U_n - \frac{\pi_{pe}^2}{16n^2\Omega_c^2}(V_n - 2U_n)^2\right]^{1/2}. \tag{21b}$$

Implicit in (20) and (18b) is a necessary condition for the instability given by

$$n^2\Omega_c^2 > k_z^2 c^2 \tag{22}$$

which is easily satisfied in gyrotrons. Equation (20) also imposes a cutoff plasma density for the instability. Physically, it is because large π_{pe} (i.e., high plasma density) tends to spoil the resonance condition (see (21a))

$$\omega_r - k_z v_{z0} - n\Omega_c \simeq 0. \tag{23}$$

On the other hand, small π_{pe}, though in favor of stronger wave-electron resonance, would also result in a low growth rate because of the small number of participating electrons. Thus, the growth rate is expected to peak at some value of π_{pe}, as is illustrated in Fig. 1 for an arbitrary set of parameters. Analytically, it can be shown from (21b) that

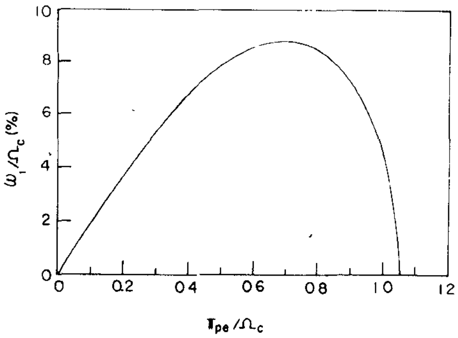


Fig. 1. ω_i versus π_{pe} as calculated from (16) for $k_z = 0$, $k_x r_{LO} = 0.6$, $\beta_{\perp 0} = 0.4$, and the fundamental harmonic mode. The growth rate generally peaks at a certain plasma frequency.

the growth rate peaks at

$$\pi_{pe}^2 = \frac{8n^2\Omega_c^2 U_n}{(V_n - 2U_n)^2}$$

and the peak growth rate is

$$\omega_i = \frac{2n\Omega_c U_n}{|V_n - 2U_n|}.$$

The maximum growth rate, as predicted by (21b), occurs at $k_z = 0$. In the limit $k_z = 0$ and

$$\pi_{pe}^2 \ll n^2\Omega_c^2 \tag{24}$$

equation (21) reduces to the following simple form:

$$\omega_r \simeq n\Omega_c \tag{25a}$$

$$\omega_i \simeq n\Omega_c \frac{\pi_{pe}}{k_x c} |J_n(k_x r_{LO})|. \tag{25b}$$

Equation (25b) can be rewritten as

$$\frac{\omega_i}{\Omega_c} = \beta_{\perp 0} \frac{\pi_{pe}}{\Omega_c} \frac{n|J_n(k_x r_{LO})|}{k_x r_{LO}} \tag{26}$$

where $\beta_{\perp 0} = v_{\perp 0}/c$. Table I lists the maximum values of $nJ_n(x)/x$ for the first ten cyclotron harmonics. One observes that, in the limit of (24), the growth rate is proportional to $\beta_{\perp 0}$ and π_{pe} and depends weakly on the harmonic number n.

Fig. 2 plots the normalized growth rate versus $k_x r_{L0}$ for several values of π_{pe}/Ω_c and $\beta_{\perp 0}$, as calculated from the full dispersion relation (16). Typically, up to 20 terms in the infinite sum are kept to ensure numerical convergence. The numerical data shown in Fig. 2 are in good agreement with the approximate solutions of (21). The growth rate exhibits a periodic dependence on $k_x r_{L0}$ as is expected from (21) and (26). This is a finite Larmor radius effect because the harmonic field component seen by a gyrating electron is a periodic function of $k_x r_{L0}$ (the ratio of the Larmor radius to the wavelength). For the fundamental harmonic modes shown in Fig. 2, the growth rate has an absolute maximum at $k_x r_{L0} = 0$ if $\pi_{pe}^2 \ll \Omega_c^2$, as is consistent with (26). But as π_{pe} increases, the maximum will shift toward finite values of $k_x r_{L0}$, showing the sensitivity of the growth rate to plasma density (cf. (21)).

IV. Monoenergetic Electrons with Velocity Spread

Electron beams used in gyrotrons are usually characterized by a distribution function of the following form:

$$f_0 = \frac{A}{2\pi\gamma_0 m_e^3 c^3}\delta(\gamma - \gamma_0)\exp\left[\frac{-(p_z - p_{z0})^2}{2\Delta p_z^2}\right] \tag{27}$$

where A is a normalization constant and Δp_z is approximately the standard deviation of p_z from the mean value p_{z0}. Equation (27) represents a uniform distribution of monoenergetic electrons with a pitch angle spread. An important aspect of the instability is that the growth rate as a function of k_z peaks at $k_z = 0$. For those modes, the resonance condition (23) is not spoiled by the velocity spread, suggesting that the growth rate is insensitive to the velocity spread.

For simplicity, we shall now restrict our consideration to the $k_z = 0$ modes. Since f_0 is a delta function of γ, the $p_\perp$-integration in (12) can be conveniently carried out if we convert the variable $p_\perp$ to γ through the following substitutions:

$$p_\perp^2 = m_e^2 c^2(\gamma^2 - 1) - p_z^2$$
$$p_\perp \, dp_\perp = m_e^2 c^2 \gamma d\gamma.$$

The resulting dispersion relation is

$$1 - \pi_{pe}^2 \sum_{n=1}^{\infty} \left[\frac{n(Q_{n-1} - Q_{n+1})}{\omega^2 - n^2\Omega_c^2} - \frac{2n^2\Omega_c^2(\omega^2 + n^2\Omega_c^2)Q_n}{k_x^2 c^2(\omega^2 - n^2\Omega_c^2)^2} \right] = 0 \tag{28}$$

where

$$Q_n \equiv \frac{A}{m_e c} \int_{-\sqrt{\gamma_0^2 - 1}\, m_e c}^{\sqrt{\gamma_0^2 - 1}\, m_e c} dp_z \exp\left[\frac{-(p_z - p_{z0})^2}{2\Delta p_z^2}\right] J_n^2(Z)$$

and

$$Z \equiv \frac{k_x}{m_e\Omega_e}\left[(\gamma_0^2 - 1)m_e^2 c^2 - p_z^2\right]^{1/2}.$$

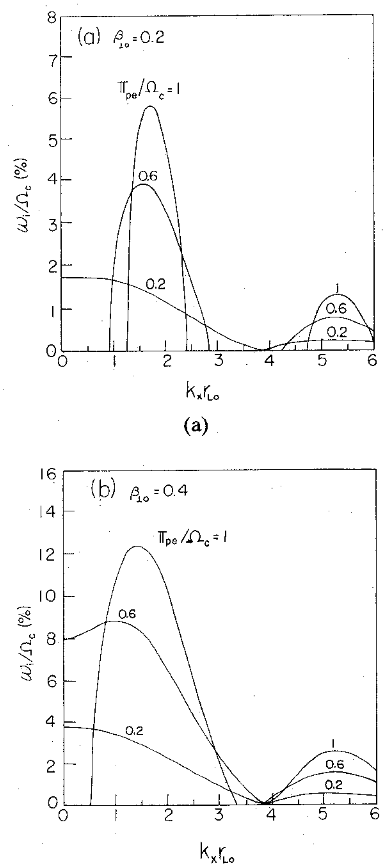
Fig. 3 shows the growth rate as a function of $k_x r_{L0}$ for $\Delta p_z/p_{z0} = 0$, 25 percent and 50 percent, where r_{L0} is the average electron Larmor radius. These data general confirm our expectation that ω_i is insensitive to the velocity spread for monoenergetic electrons. In fact, in regions where the "cold"-beam growth rates are small, velocity spread can enhance the growth rates. According to computer simulated performance [8], electron beams used in gyrotrons normally have $\Delta p_z/p_{z0} < 20$ percent. For such beams, the cold-beam dispersion relation (16) gives a fairly good estimate of the growth rate.

V. Finite Geometry Effect

We consider an infinite plasma slab that is confined in the region between $x = 0$ and $x = d$. As before, the plasma density and the external magnetic field ($B_0 \underset{\sim}{e}_z$) are uniform.

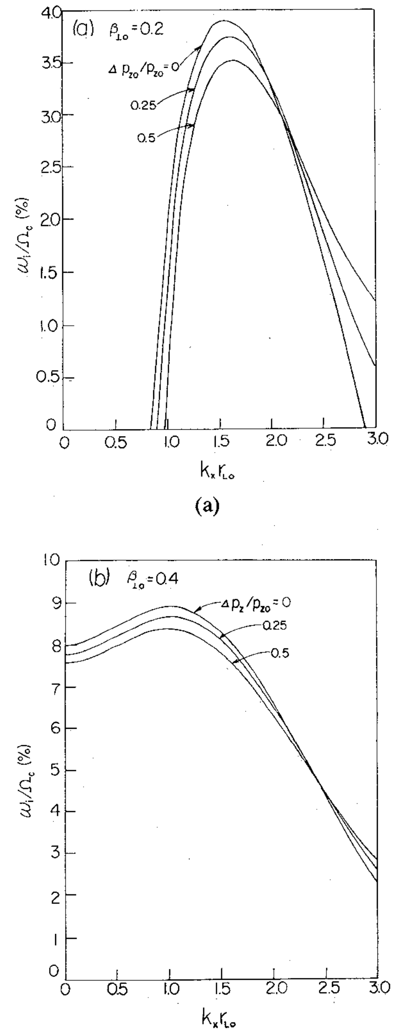
TABLE I
MAXIMUM VALUES OF $nJ_n(x)/x$

n	1	2	3	4	5	6	7	8	9	10
x	0	2.3	3.6	4.8	6.0	7.1	8.2	9.3	10.4	11.4
$\frac{nJ_n(x)}{x}$	0.500	0.360	0.332	0.315	0.302	0.291	0.282	0.274	0.267	0.261



(b)

Fig. 2. ω_i versus $k_x r_{L0}$ as calculated from (16) for $k_z = 0$ and the fundamental harmonic mode. The growth rate is a sensitive function of $k_x r_{L0}$ and π_{pe}.



(b)

Fig. 3. ω_i versus $k_x r_{L0}$ as calculated from (28) for $k_z = 0$, $\pi_{pe}/\Omega_c = 0.6$, and the fundamental harmonic mode. The growth rate is insensitive to electron velocity spread.

The dispersion relation treated in the previous sections is still the condition for nontrivial solutions, but there are now boundary conditions to be satisfied at $x = 0$ and $x = d$. For simplicity, we again limit our consideration to the most unstable modes ($k_z = 0$). Since the wave field is independent of y and z and there is no net charge in the slab, Gauss law requires that the wave electric field vanish on the boundaries as well as outside. Given the assumed sinusoidal dependence of the wave (wavelength $= 2\pi/k_x$), boundary conditions require k_x to have discrete values

$$k_x = \frac{m\pi}{d}, \qquad m = 1, 2, \cdots. \tag{29}$$

It is interesting to note that for $k_y = k_z = 0$ as assumed in this section, the mode resides completely within the beam. The beam behaves much like an "electrostatic resonator."

This phenomenon illustrates a basic difference between the gyrotron and the conventional linear microwave tubes. In the linear microwave tubes, the free energy (streaming energy in the axial direction) exists only when there is a circuit which provides a stationary frame of reference. That is, in the absence of a circuit or even in the presence of a circuit that is uniform in the axial direction, the free energy vanishes in the reference frame of the beam. Consequently, all unstable modes must be coupled with an axially nonuniform circuit as in the helix TWT or the klystron. In contrast, the free energy of a gyrotron (gyrational energy of electrons) exists independent of the circuit. Hence, the beam can sustain an instability by itself (e.g., the electrostatic cyclotron instability) or drives an instability on the modes of a smooth waveguide (e.g., the gyro-TWT).

VI. Application to Gyrotrons

A major implication of the instability concerns the amplification of electrostatic beam noises in gyrotrons. Consider, for example, the *Ka*-band gyro-TWT experiments reported by Barnett *et al.* [9], [10]. The experimental parameters are $\gamma_0 \simeq 1.137$ (70 keV), $v_{\perp 0}/c \simeq 0.396$, $v_{z0}/c \simeq 0.264$, $I \simeq 3$ A, $B_0 \simeq 13$ kG, $r_{L0} \simeq 0.06$ cm, and $r_0 \simeq 0.25$ cm, where I is the beam current and r_0 is the average radius of the hollow beam. The thickness of the beam annulus is approximately $3r_{L0}$, as inferred from computer simulations [8]. From these parameters, we find

$$k_x r_{L0} \simeq m\pi/3, \qquad m = 1, 2, \cdots$$

$$\pi_{pe} \simeq 4.84 \times 10^9 \text{ rad/s}$$

and

$$\Omega_c \simeq 2.01 \times 10^{11} \text{ rad/s}$$

where π_{pe} is calculated on the basis of average electron density. Since $\pi_{pe} \ll \Omega_c$, (26) may be used. To estimate the growth rate for the $n = 1$ mode, we use the minimum $k_x r_{L0}$ ($\simeq \pi/3$). Hence, $J_1(k_x r_{L0})/k_x r_{L0} \simeq 0.43$. For the $n > 1$ modes, we use the values of $nJ_n(x)/x$ as tabulated in Table I. Equation (26) then gives

$$\frac{\omega_i}{\Omega_c} \times 1000 \simeq 4.16, 3.49, 3.21, 3.05, \text{ and } 2.92 \qquad (30)$$

for the lowest five harmonics, respectively.

The rate of noise amplification g is related to the growth rate by

$$g = 8.7\omega_i/v_{z0} \text{ dB/unit length} \qquad (31)$$

Thus

$$g \simeq 0.91, 0.77, 0.71, 0.67, \text{ and } 0.65 \text{ dB/cm} \qquad (32)$$

for the lowest five harmonics, respectively. The measured RF gain is approximately 1 dB/cm.

As another example, we consider the *C*-band gyro-TWT experiment reported by Symons *et al.* [11]. The parameters of the experiment are: $\gamma_0 \simeq 1.117$ (60 keV), $v_{\perp 0}/c \simeq 0.371$, $v_{z0}/c \simeq 0.247$, $I \simeq 5$ A, $B_0 \simeq 2$ kG, $r_{L0} \simeq 0.35$ cm, and $r_0 \simeq 0.9$ cm. Similarly, assuming a beam thickness of $3r_{L0}$, we obtain

$$k_x r_{LO} \simeq m\pi/3, \qquad m = 1, 2, \cdots,$$

$$\pi_{pe} \simeq 1.42 \times 10^9 \text{ rad/s}$$

and

$$\Omega_c \simeq 3.15 \times 10^{10} \text{ rad/s}.$$

Equations (26) and (31) then give

$$\frac{\omega_i}{\Omega_c} \times 1000 \simeq 7.19, 6.02, 5.55, 5.27, \text{ and } 5.05$$

and

$$g \simeq 0.27, 0.22, 0.21, 0.20, \text{ and } 0.19 \text{ dB/cm} \qquad (33)$$

for the lowest five harmonics, respectively. The measured RF gain is approximately 0.5 dB/cm.

We note that (16) is valid under the condition $k_x^2 c^2 \gg \omega_{pe}^2$ and $\omega/k_z \gg v_z$. Equation (26) is valid under the additional conditions $\omega_r \gg \omega_i$, $k_x \gg {}_{kz}$, and $n^2\Omega_e^2 \gg \omega_{pe}^2$. All these conditions are well satisfied for the examples examined. The neglect of the dc space charge effect (i.e., the assumption of a neutral plasma) is also justified under the condition $\Omega_c^2 \gg \omega_{pe}^2$. However, when applying (26) to a hollow beam of diffusive density profile, the assumption of uniform beam density is justifiable only when

$$k_x d \gg 1 \qquad (34)$$

that is, the beam density does not vary much over one transverse wavelength. Condition (34) is satisfied in our estimates of the growth rates for the $n > 1$ modes, but not for the $n = 1$ mode. Had we assumed a larger $k_x r_{L0}$ for the $n = 1$ mode, the growth rate would have been smaller (see Fig. 2). Hence, the gain values for the $n = 1$ mode as shown in (32) and (33) are somewhat exaggerated.

In both examples, the noise gain is comparable to the RF gain. As shown in Section IV, the instability is not easily stabilized by the beam velocity spread. Furthermore, the most unstable modes are shown in Section V to be uncoupled to the circuit. This is yet another reason that the instability is difficult to stabilize (e.g., by a resistive wall). It may be concluded that the electrostatic cyclotron instability can lead to significant noise enhancement in gyrotrons.

The electrostatic cyclotron waves, when built up to high levels, may adversely affect the saturated power of gyrotrons as well. Efficiencies achieved in gyro-TWT's are known to be significantly lower than theoretical predictions [12]. Although there could be numerous reasons for the low efficiency, the examples shown here suggest that interference from the electrostatic modes might be an important contributing factor. It is possible that these modes have escaped detection because of their weak coupling to the circuit.

VII. Comparison with Previous Works

There are many early works that deal with various aspects of the electrostatic cyclotron modes, but not in the context of gyrotrons. A brief review of these works can be found in Bekefi [6, p. 234]. In short, the works cited by Bekefi employ a nonrelativistic treatment. It is nevertheless interesting to compare the relativistic dispersion relation (16) with its nonrelativistic counterpart [6, eq. (7.24)]. In the absence of the relativistic term as underlined in (16), the two equations are identical. Without the source term for the instability, the nonrelativistic dispersion relation was therefore found to be stable [6, p. 234]. A later paper by Blanken *et al.* [1] studied the electrostatic cyclotron modes with a relativistic formalism. The interest there was to investigate the second cyclotron harmonic emission observed in a magnetic mirror. Although the instability is of the same type, the results of [1], numerically calculated for a loss cone distribution and limited to the second cyclotron harmonic, are not applicable to gyrotrons.

Charbit *et al.* [2] have derived a space-charge instability on the gyrotron beam and pointed out its potential role as a noise amplifying mechanism. The space-charge instability in [2] can be recovered from (21) in the long wavelength and/or zero Larmor radius limit. To show this, we take the limit $k_x r_{L0} \to 0$ and let $k_z = 0$, then only the $n = 1$ mode remains and (21) reduces to

$$\omega_r \simeq \Omega_c\left(1+\pi_{pe}^2/2\Omega_c^2\right)^{1/2}$$

$$\simeq \Omega_c\left(1+\pi_{pe}^2/4\Omega_c^2\right) \qquad (35a)$$

$$\omega_i \simeq \pi_{pe}\left(\beta_{\perp 0}^2/4-\pi_{pe}^2/16\Omega_c^2\right)^{1/2} \qquad (35b)$$

where the last equality in (35a) is consistent with the assumption $(\omega_r - \Omega_c)/\Omega_c \ll 1$ made in Charbit *et al.* Equation (35a) and (35b) are precisely (27) in Charbit *et al.* In a gyrotron, electrons necessarily have finite Larmor radius, Charbit's result is then restricted to macroscopic (long wavelength) modes at the fundamental cyclotron harmonic frequency. For study of broad-band microscopic noises, (21) is more appropriate.

Hirshfield [3] has investigated the harmonics of the electrostatic cyclotron instability and suggested that they could be the basis for a harmonic cyclotron maser (see next section). In Hirshfield's study, the emphasis is on the coupling between the nearly electrostatic cyclotron modes of the electron beam and the electromagnetic modes of a resonator. The growth rate for the electrostatic modes [3, eq. (4)], calculated under the assumptions $k_z = 0$ and $\pi_{pe}^2 \ll \Omega_c^2$, is identical to (26), which was obtained from (21) in the limits $k_z = 0$ and $\pi_{pe}^2 \ll n^2\Omega_c^2$.

In summary, the results quoted from [6, p. 234], [2], and [3] can be unified by the dispersion relation in (16).

VIII. Discussion

The derivation of (11) is a straightforward exercise of substituting the standard plasma dielectric tensor (10) in a general dispersion relation (8). In (11), the relativistic effects are implicitly contained. Subsequent integrations by parts, however cast (11) in a form (12) which exhibits the relativistic effects explicitly. When we specialize to the distribution function of (15), the dispersion relation (16) is found to differ from its nonrelativistic counterpart [6, eq. (7.24)] by only one extra term. An analogy can be found in the cyclotron maser instability where the relativistic dispersion relation differs from the nonrelativistic version also by one extra term [13].

Finally, we comment on the possibility of generating electromagnetic radiation through the electrostatic cyclotron instability. It is speculated in [2] that the electrostatic cyclotron instability might be useful in a gyroklystron by permitting a high gain with a short distance between cavities. A different possibility was suggested by Hirshfield [3], who noted that, at high cyclotron harmonics, the electrostatic cyclotron instability has a far greater growth rate than the cyclotron maser instability, and hence, can be exploited to generate very-high-frequency electromagnetic radiation. Experiments by Ebrahim *et al.* [14] have indeed detected radiation up to the fourth cyclotron harmonic at a low efficiency.

In the idealized model in Section V, the electrostatic cyclotron instability is completely localized within the beam and, therefore, provides no means for coupling to the electromagnetic circuit wave. From this perspective, the coupling between electrostatic and electromagnetic waves depends on the "violation" of the assumptions made in the idealized model. When the condition (1) for the electrostatic approximation, $k_x^2 c^2 \gg \omega_{pe}^2$, is violated, there will be a magnetic wave component and some coupling to the circuit wave. Table I together with (26) shows that k_x for the maximum growth rate assumes increasingly greater values as the harmonic number increases. Consequently, the possibility of having a small magnetic wave component through violation of condition (1) decreases with increasing harmonic number. This may be a qualitative reason why only the first few harmonics have been observed in [14], while calculated start-oscillation currents are the same for all harmonics. A second point concerns the assumption $k_y = k_z = 0$ made in Section V. When this condition is violated, as it must be in reality, there will be an oscillating fringe electric field (hence, the induced magnetic field) on the beam boundary, which also provides a way for coupling to the circuit wave. In Hirshfield's study [3], for example the coupling is due to $k_y \neq 0$. For either $k_y \neq 0$ or $k_z \neq 0$, however, the wave in the beam is still predominantly electrostatic as long as $(k_x^2 + k_y^2)c^2 \gg \omega_{pe}^2$. As the beam leaves the circuit, it carries with it the electrostatic wave energy. So it appears that the saturated efficiency may be intrinsically low unless the radiated power coupled out of the beam far exceeds what is lost in the electrostatic waves.

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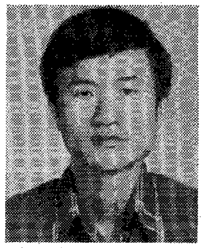

Kuan Ren Chen was born in Taiwan in 1960. He received the B.S. and M.S. degrees in nuclear engineering from National Tsing Hua University, Taiwan, Republic of China, in 1982 and 1984, respectively. His thesis work was on propagation of plasma waves.

In the summer of 1984, he served as a Research Assistant in the Department of Physics, National Tsing Hua University. He is presently serving in the Reserve Officers' Training Corps.

Kwo Ray Chu (SM'82) received the B.S. degree in physics from National Taiwan University in 1965, the M.S. degree in physics from the University of Massachusetts in 1968, and the Ph.D. degree in applied physics from Cornell University in 1972. He specializes in microwave theory, relativistic electronics, plasma physics, and controlled thermonuclear fusion.

From April 1973 to September 1977, he was a Research Physicist with Science Applications, Inc. From September 1977 to September 1983, he was with the U.S. Naval Research Laboratory, serving as Head of the Advanced Concepts Section in the High Power Electromagnetic Radiation Branch. During this period, he served for three years as Adjunct Associate Professor of Applied Physics at Yale University. Since September 1983, he has been a Visiting Professor in the Department of Physics, National Tsing Hua University, Taiwan, Republic of China. He has also been a Consultant to Science Applications International Corporation since September 1983.

He was elected a Fellow of the American Physical Society in 1983.